Geometrical and Statistical Properties of Evolution Equations on Singular Spaces.

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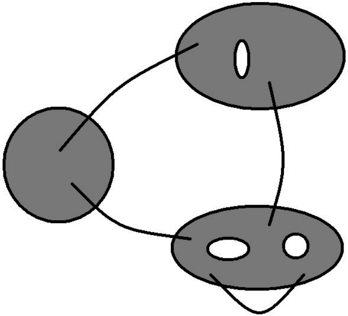
Outline

Singular spaces (decorated graphs)

Decorated graph Γ — topological space, obtained from a graph via replacing vertices by smooth manifolds M_k, $\dim M_k \leq 3$.



Spectral theory:

L.D. Faddev, B.S. Pavlov, Theor. Math. Phys, 1983, 55(2), 257-268.

B.S. Pavlov, Theor. Math. Phys, 1984, 59(3), 345-353.

P.Exner, P. Šěba.J. Math. Phys. 28 (1987), 386-391.

J. Brüning and V.A. Geyler, M. Lobanov, S. Roganova, A. Tolchennikov, etc.

Laplacian

Edges γ_j of the graph are smooth parametrized curves, M_k are complete Riemannian manifolds.

Definition of the Laplace operator Δ: 2 conditions.

- Δ is self-adjont;
- If M is a disconnected then

$$\Delta = \oplus_j \frac{d^2}{dz^2} \oplus_k \Delta_k.$$

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Laplacian

Formal definition. Consider the direct sum

$$\Delta_0 = \oplus_j \frac{d^2}{dz_j^2} \oplus_k \Delta_k$$

with the domain

$$H^2(\Gamma) = \oplus_j H^2(\gamma_j) \oplus_k H^2(M_k).$$

Δ is a self-adjoint extension of the restriction $\Delta_0|_L$, where

$$L = \{\psi \in H^2(\Gamma), \quad \psi(q_s) = 0\}.$$

Coupling conditions

Boundary conditions: on the edges consider $\psi(q_j)$, $\psi'(q_j)$. On the manifolds functions have singularities of the Green function type

$$\psi = a_j F(x) + b_j + o(1),$$

$$F = \begin{cases} -\dfrac{c}{2\pi}\log\rho, & \dim M = 2; \\ \dfrac{c}{4\pi\rho}, & \dim \mathrm{M} = 3. \end{cases}$$

Coupling conditions

Vector $\xi = (u, v)$, $u = (\psi'(q_1), \ldots, \psi'(q_N), a_1, \ldots, a_N)$,
$v = (\psi(q_1), \ldots, \psi(q_N), b_1, \ldots, b_N)$.
In $\mathbb{C}^{2N} \oplus \mathbb{C}^{2N}$ consider standard skew-Hermitian form

$$[\xi^1, \xi^2] = \sum_{j=1}^{2N} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2).$$

and fix the Lagrangian plane L.
Coupling conditions

$$\xi \in L, \quad -i(E+U)u + (E-U)v = 0,$$

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Local coupling conditions — for each point separately:

$$L = \oplus_q L_q$$

.

For each point of gluing q $\xi_q = (u_q, v_q)$, $u_q = (\psi'(q), a)$, $v_q = (\psi(q), b)$,

$$-i(E + U_q)u_q + (E - U_q)v_q = 0,$$

U_q is a unitary 2×2-matrix.

Example: wave equation on the hybrid space

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The hybrid space $\mathbb{R}^3$ with a half-line $\mathbb{R}_+$, coupled at the point $z = 0$, $x = 0$.

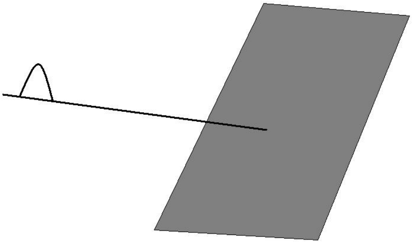

Figure: Hybrid space

Wave equation

Wave equation

$$\psi_{tt} = \Delta\psi$$

$$\psi|_{t=0} = \psi_0(z), \quad \psi_t|_{t=0} = \psi_0'(z), \quad \psi_0 \in C_0^\infty(\mathbb{R}_+).$$

Solution

$$\psi = \begin{cases} \psi_0(t+z) + \chi(t-z), & z \in \mathbb{R}_+, \\ \dfrac{f(t-|x|)}{|x|}, & x \in \mathbb{R}^3. \end{cases}$$

Functions $\chi(z)$, $f(z)$ can be expressed explicitly in terms of ψ_0 and coupling matrix U.

Examples

Complete reflection $f = 0$.

$$U = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad L = L_1 \oplus L_2.$$

$$\chi(z) = \frac{-i\frac{\partial}{\partial z}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{-i\frac{\partial}{\partial z}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}\psi_0(z).$$

Examples

Complete transmission $\chi = 0$

$$U = \frac{1}{1+4\pi}\begin{pmatrix} 1-4\pi & 4\sqrt{\pi}e^{i\varphi} \\ -4\sqrt{\pi}e^{-i\varphi} & 1-4\pi \end{pmatrix}.$$

$$f(z) = \frac{ie^{i\varphi}}{2\sqrt{\pi}}\psi_0(z).$$

Further examples (Anna V. Tsvetkova): explicit formulas for the wave equation on a space of constant curvature (S^2, S^3, H^2, H^3) coupled to the half-line.

Outline

Gaussian packet

Let Γ be a decorated graph.
Cauchy problem for Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\Delta\psi + V\psi$$

$$\psi|_{t=0} = A_0 e^{\frac{iS_0(z)}{h}}.$$

$$S_0 = p_0(z-z_0) + \frac{1}{2}Q_0(z-z_0)^2, \quad \Im p_0 = 0, \quad \Im Q_0 > 0.$$

Asymptotics as $h \to 0$; narrow Gaussian packet (squeezed state) located on the edge.
Certain technical conditions on V, metrics and S_0.

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Solution for small t

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Assertion

$$\psi(z,t,h) = A(t)e^{\frac{iS(z,t)}{h}} + O(\sqrt{h}).$$

$$S(z,t) = \sigma(t) + P(t)(z - Z(t)) + \frac{1}{2}Q(t)(z - Z(t))^2.$$

$(P(t), Z(t))$ is the solution of the classical Hamiltonian system

$$\dot{z} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial z}, \quad H = \frac{1}{2}p^2 + V(z).$$

$$P(0) = p_0, \quad Z(0) = z_0.$$

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Outline

Scattering on a manifold

Let Γ be a half-line, connected with the manifold M in a single point q.

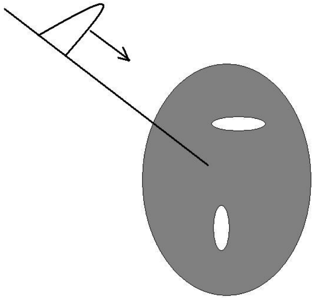

Figure: Scattering on a manifold

Scattering on a manifold

Theorem

For certain time interval solution has the form

$$\psi = \begin{cases} A(t)e^{\frac{iS(z,t)}{h}}, & z \in \mathbb{R}_+, \\ K_{\Lambda_t}[B(x,t)], & x \in M. \end{cases} + O(\sqrt{h})$$

K is the Maslov canonic operator on the isotropic manifold Λ_t with complex germ. A and B can be expressed explicitly in terms of the coupling matrix U.

Scattering on a manifold

Support of the solution on M.

At the instant of scattering t_0 consider the sphere in T_q^*M: $\Lambda_0 : |p| = |P(t_0)|$. Consider the flow g_t of the classical Hamiltonian system on M with the Hamiltonian $H = \frac{1}{2}|p|^2 + V$. Isotropic manifold $\Lambda_t = g_t\Lambda_0$; as $h \to 0$ $supp\psi \to \pi(\Lambda_t)$, π is the natural projection.

In general position ψ is localized near the surface of codimension 1 (Gaussian packet near the hypersurface). For $V = 0$ the support coincides with the geodesic sphere.

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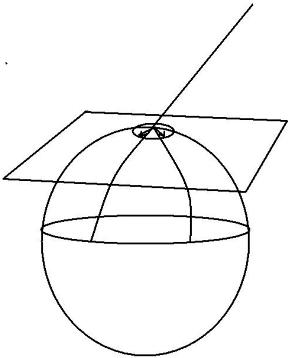

Figure: Support of the solution

Outline

Number of the Gaussian packets

Let Γ be a decorated graph with finite edges. For arbitrary finite t the solution has the form

$$\psi = \sum_j \psi_j + O(\sqrt{h}),$$

where ψ_j are the Gaussian packets.
Let $N(t)$ be the number of packets, localized on the edges of Γ (not on the manifolds).
Let t_j be times of passage of the trajectories along the edges of the graph and between gluing points on the manifolds.
Assumption: there is a finite number of times $t_1, \dots, t_M$.

Statistics of Gaussian packets

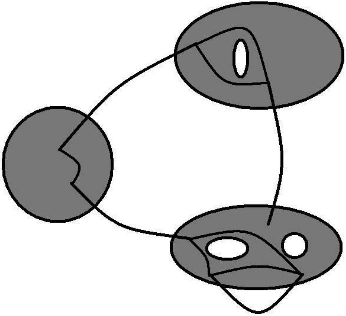

Figure: Times of connection

Statistics of Gaussian packets

Theorem

Let t_j be linearly independent over $\mathbb{Q}$. As $t \to \infty$

$$N(t) = Ct^{M-1} + o(t^{M-1}).$$

Constant C

Theorem

For almost all t_j

$$C = \frac{\sum_{edges} t_j}{2^{N-2}(M-1)! \prod_{all} t_j}.$$

Here N — number of points of gluing, M — number of times t_j.

Distribution of the packets

Uniform distribution.
Let δ be a segment on arbitrary edge of Γ.
N_δ is a number of packets, located on δ.

Theorem

$$\lim_{t\to\infty}\frac{N_\delta(t)}{N(t)}=\frac{t_\delta}{\sum_j t_j}.$$

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Theorem

$$\lim_{t\to\infty} \frac{N_\delta(t)}{N(t)} = \frac{t_\delta}{\sum_j t_j}.$$

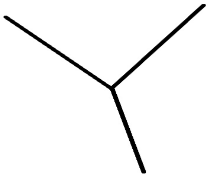
Ideas of the proofs.

1. Reduction to the problem of statistic of classical particles on metric graphs.
2. Application of the previous results (V. Chernyshev, A.Sh.) obtained for the case of metric graphs.
3. Reduction to the certain problem of analytic number theory - counting the number of integral points in a large simplex.

Hardy

Example: Star with three edges — Hardy - Littlewood formula (1927).

$$N(t) = \frac{t^2}{8}\frac{t_1 + t_2 + t_3}{t_1 t_2 t_3} + \frac{t}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) + o(t).$$



Outline

The cylinder with the segment

Examples of spaces with infinite number of times (V. Chernyshev, A. Tolchennikov)

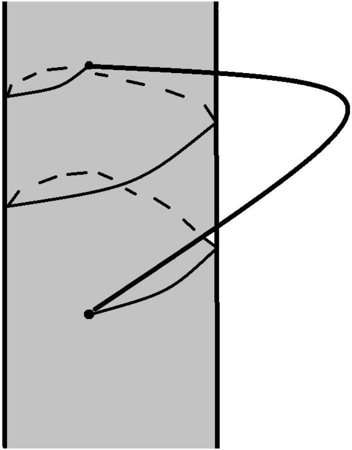


Figure: The segment glued to the cylinder in two points

$$N(t) = C\exp\left(\sqrt{\frac{2}{3R}}\pi\sqrt{t}(1+o(1))\right).$$

Here R is the radius of the cylinder.

Flat torus with the segment

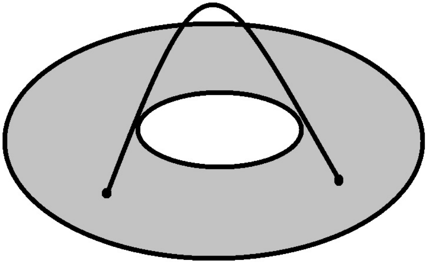


Figure: The segment glued to the torus in two points

$$N(t) = C\exp\left(3\sqrt[3]{\frac{5\pi}{8ab}\zeta(3)}t^{2/3}(1+o(1))\right).$$

Here a, b are lengths of the fundamental cycles of the torus.
For 3D torus

$$N(t) = C\exp\left(4\sqrt[4]{\frac{\pi}{3abc}\zeta(4)}t^{3/4}(1+o(1))\right).$$

The proof uses the following result.
Theorem (V.E. Nazaikinskii, 2013). Let N(t) be the number of non-negative integer solutions of inequality $\sum_{i=1}^{\infty} \lambda_i N_i \leq t$ and for sequence λ_j a counting function $\rho(\lambda) = \sharp(j|\lambda_j \leq \lambda)$ has asymptotics

$$\rho = c_0 \lambda^{1+\gamma}(1 + O(\lambda^{-\varepsilon})), \quad \varepsilon > 0.$$

Then

$$\log N(t) = (\gamma+2)\left(\frac{c_0 \Gamma(\gamma+2)\zeta(\gamma+2)}{(\gamma+1)^{\gamma+1}}\right)^{\frac{1}{\gamma+2}} t^{\frac{\gamma+1}{\gamma+3}}(1 + o(1)).$$

as t goes to infinity