# Introduction to the Theory of Integrable Systems

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1. The Lagrangian formalism: elements of variational calculus, Euler–Lagrange equation, Lagrangian approach to Newtonian mechanics, variational principle and geodesics, Noether's theorem, generalized variational problem with higher-order derivatives.

2. The Hamiltonian formalism: Hamiltonian equations, Hamiltonian form of Lagrangian equation, Poisson brackets and first integrals.

3. Symplectic and Poisson manifolds, Darboux's theorem. Hamiltonian vector fields. Symplectic leaves, Casimir functions.

4. Liouville integrability: Liouville–Arnold theorem and action-angle variables.

5. Classical examples of integrable systems: Kepler problem, the Lagrange top, geodesics on an ellipsoid.

6. Lax representation and first integrals. Lax representations with spectral parameter.

7. The theory of one-dimensional Toda lattice: Lax representation, Liouville integrability, inverse scattering method. Relation to the $QR$-algorithm.

8. Bi-Hamiltonian approach and Lenard–Magri scheme.

9. The theory of Veselov–Shabat dressing chain: Darboux transformations, Liouville integrability and relation to the Painvelé equations.

10. Integrable discrete equations on quad-graphs: 3D-consistency and zero curvature representation. Cauchy problem and the Adler–Bobenko–Suris theorem.

11. Isospectral deformations of the Schrödinger operator and the Korteweg-de Vries equation (KdV). Single soliton solution.

12. Gelfand–Dickey approach: pseudo-differential operators and the square root of the Schrödinger equation. The KdV hierarchy.

13. Elements of the scattering theory for one-dimensional Schr$\ddot{\text{d}}$inger operator with rapidly decaying potential.

14. The inverse scattering method for the KdV equation: Gelfand–Levitan–Marchenko eqution, Gardner–Green–Kruskal–Miura equations.

15. Reflectionless potentials and multi-soliton solutions for the KdV equation. Interaction of solitons. Asymptotics of solutions to the KdV equation.

16. Modified KdV equation, the Miura transformation. Bäcklund transformations for the KdV equation. The Hirota's method.

17. Gardner–Zakharov–Faddeev bracket. Hamiltonian and bi-Hamiltonian structures for the KdV equation.

18. Polynomial integrals of motion for the KdV equation and its complete integrability.

19. Asymptotic curves on the surfaces of constant negative curvature and the sin-Gordon equation.

20. Differential geometry of the hydrodynamic type systems.
21. The Liouville equation and Darboux integrable hyperbolic equations.
22. Darboux–Laplace transformations and two-dimensional Toda lattices.