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Nonsymmetric Forms of Envelopes of Relativistic Magnetic Stars[1]

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Abstract—Within the framework of a previously proposed relativistic model of a magnetized perfectly conducting material surface without internal stresses, an axisymmetrical equilibrium problem is considered of the rotation of a stellar envelope, supported by a radial hypersonic stellar wind in a given central gravitational field and dipole magnetic field. In the absence of either the magnetic field or the stellar wind, the equations have an explicit solution; in the general case the solution is obtained numerically. Of special interest are the envelope shapes asymmetric with respect to the equator (relative to the direction of the dipole), which have the greatest stability, in particular, bottle-shaped structures.

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1. INTRODUCTION

The photographs of planetary nebulae, recently obtained with the help of high-resolution telescopes, have revealed the existence of different layered structures consisting of gas-dust envelopes around stars [1].

In connection with the investigation of astrophysical objects with strong magnetic field, the analysis of stationary spatial configurations of thin conductive envelopes, apart from the conventional theory of flat plasma discs [2], is of interest. The possible (and, in some cases, even dominating) action of stellar wind should also be taken into account.

A study of the shapes of such relativistic objects as neutron stars, galactic nuclei, white and black holes is also of interest. In the Newtonian approximation the problem is considered in [3, 4].

2. EXTERNAL FIELDS

For the description of the electromagnetic field of a massive magnetic dipole in general relativity, the Bonnor solution to the electrovacuum equations with two Killing vectors [5] can be used, but it is easier to construct anew the field of a magnetic dipole in the Schwarzschild metric

$$ds^2 = (1 - 2m/r)\,dt^2 - (1 - 2m/r)^{-1}\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

where m is the mass of a star; the speed of light c and the gravitational constant G are taken to be equal to unity.

The corresponding solution of the Maxwell equations

$$\nabla_j F^{ij} = 0, \quad F_{ij} = \nabla_i A_j - \nabla_j A_i$$

for the vector potential A_i of the electromagnetic field ($i = 0,\ 1,\ 2,\ 3$) in the Schwarzschild metric has the form

$$A_\varphi = -\frac{3M\sin^2\theta}{4m^3}\left(m^2 + \frac{r^2}{2}\ln(1 - 2m/r) + mr\right),$$

where M is the magnitude of the dipole. The remaining components of A_i are zero. In solving the problem of dynamical equilibrium of a material envelope we restrict ourselves to the domain $r > 2m$.

The components of the tensor F_{ij} are

$$F_{r\varphi} = -\frac{3M\sin^2\theta}{4m^3}\left(r\ln(1 - 2m/r) + 2m\frac{r-m}{r-2m}\right),$$

$$F_{\theta\varphi} = -\frac{3M\sin\theta\cos\theta}{2m^3} \times\left(m^2 + \frac{r^2}{2}\ln(1 - 2m/r) + mr\right).$$

The stellar wind is considered to be stationary, spherically symmetric and hypersonic. This assumption is connected with the neglect of the wind's thermal pressure and, possibly, if the wind is a conducting plasma, of the Lorentz force exerted on the wind. The action of a stellar wind with the rest mass density ρ and 4-velocity w^i is determined by the equations

$$\nabla_i(\rho w^i) = 0, \quad w^k\nabla_k w^i = 0, \quad w^i w_i = 1,$$

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whose solution in the metric (1) yields

$$\rho w^r = \frac{Q}{4\pi r^2}, \quad w^r = \left(\frac{w_\infty^2}{1-w_\infty^2} + \frac{2m}{r}\right)^{1/2},$$
$$w^t = \frac{1}{(1-2m/r)\sqrt{1-w_\infty^2}},$$

where Q is the strength of the wind source and w_∞ is the radial 3-velocity at infinity.

3. EQUATIONS OF AN ENVELOPE

Let the spatial part of the equations of motion for a cold envelope have the form

$$\left(\sigma u^\alpha \nabla_\alpha u^i + j_k F^{ik} - k\rho w^i w^k n_k\right)_{\perp l^i} = 0,$$
$$\nabla_\alpha(\sigma u^\alpha) = 0, \quad 4\pi j_i = [F_{ij}]n^j. \qquad (1)$$

Here σ is the surface density of the envelope rest mass, u^i is the 4-velocity, n^i is the 4-normal to the envelope, $n^i n_i = -1$, $u^i u_i = 1$, $n^i u_i = 0$; the derivative ∇_α ($\alpha = 0,\ 1,\ 2$) is connected with the internal geometry of the envelope. The jump $[F_{ij}]n^j$ corresponds to a self-frozen-in magnetic field of the envelope, k is the envelope's momentum absorption coefficient. Absorption of rest mass by the envelope is neglected. The subtler question of energy absorption is also left out of our consideration; the vector of the observer's reference frame is $l^i = \delta^i_t$.

The derivation of the equations is connected with a study of the three-dimensional structure of a thin envelope, when the internal magnetic field is supposed to be linearly distributed across the width of the envelope and is being continuously sewn together with the external field, which obeys the Maxwell equations and, in turn, slightly differs from the given field of the central dipole.

By virtue of an infinite conductivity inside the envelope, the equality $u^i F_{ij} = 0$ holds. This leads to independence of the Lagrangian variables $\hat{F}_{\mu 3}$, where $\mu = 1, 2$ are longitudinal indices, and hence the variables' jump is time-independent.

4. STATIONARY ROTATION OF THE ENVELOPE

Let us consider the problem of stationary rotation of an axisymmetric envelope with azimuth current. Let the 4-velocity of the envelope have the form

$$(u^i) = \frac{(1, 0, 0, \omega)}{\sqrt{1 - 2m/r - r^2\sin^2\theta\,\omega^2}},$$

where the functions $r = r(\theta)$ and $\omega = \omega(\theta)$ determine the shape of the envelope and the distribution of its angular 3-velocity, respectively. The surface density σ and the component j_φ of the current are also functions of the angle θ.

The vector of normal is

$$(n_i) = \frac{(0, 1, -r', 0)}{\sqrt{1 - 2m/r + r'^2/r^2}}.$$

In the stationary axisymmetric case, the continuity equation as well as the φ projection of Eq. (1), together with the aforementioned freezing-in conditions (Section 3) are satisfied identically. There remain two equations involving four functions of the angle θ, so in this case, of course, different statements of the problem are possible. Let us dwell on the following one. We shall consider σ and ω to be constant.

If we introduce the dimensionless parameters

$$\mu = w_\infty^2/c^2, \quad \delta = \frac{\omega^2 r_g^3}{Gm}, \quad \varepsilon = \left(\frac{kQc}{4\pi G\sigma m}\right)^2,$$

the length scale $r_g = 2Gm/c^2$, the dimensionless variable $r_1 = r/r_g$ and the quantity $j_1 = 3j_\varphi M/(r_g^5\omega^2\sigma c)$ (further on the index "1" is omitted), then the equations for envelope's shape assume the form

$$\frac{j}{r^4} f_1(r) + \frac{2r\sin^2\theta}{2r - 2 - \delta r^3\sin^2\theta} = 0, \qquad (2)$$

$$1 - \frac{1}{r} + \frac{r'^2}{r^2}$$
$$= \frac{\varepsilon(\mu/(1-\mu) + 1/r)}{(1-1/r)^2}\left(1 - \frac{1}{r} - \frac{\delta_2}{2}r^2\sin^2\theta\right)^2$$
$$\times \frac{f_1^2(r)}{\left[(1-\delta r^3\sin^2\theta)f_1(r) - 2\delta r^4\sin^2\theta f_2(r)\right]^2}, \qquad (3)$$

where

$$f_1(r) = 1 + 2r^2\ln(1 - 1/r) + 2r,$$
$$f_2(r) = r\ln(1 - 1/r) + \frac{r - 1/2}{r - 1}.$$

Eq. (2) is the θ-projection of Eq. (1); Eq. (3) is obtained by eliminating j from the radial equation.

5. EXACT SOLUTIONS

A. In the absence of rotation ($\delta = 0$), Eq. (3) assumes the form

$$r' = \pm r\left(\frac{\varepsilon\mu}{1-\mu} - 1 + \frac{1+\varepsilon}{r}\right)^{1/2}$$

and has the following family of solutions:

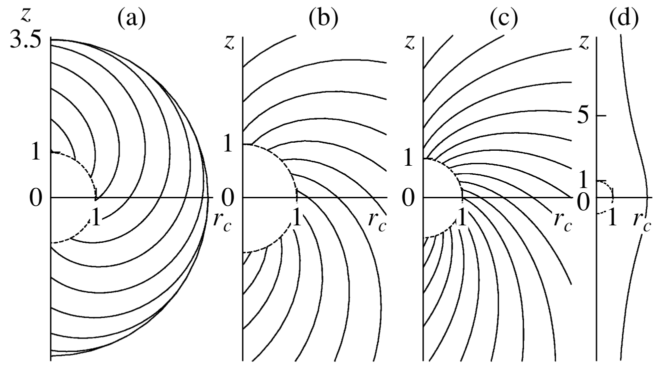

Fig. 1. Nonrotating envelopes ($\delta = 0$; (a) $\mu = 0.3$, $\varepsilon = 1$, (b) $\mu = 0.5$, $\varepsilon = 1$, (c) $\mu = 0.7$, $\varepsilon = 1$) and a double bottle in the absence of a stellar wind ((d) $\varepsilon = 0$, $\delta = 0.1$).

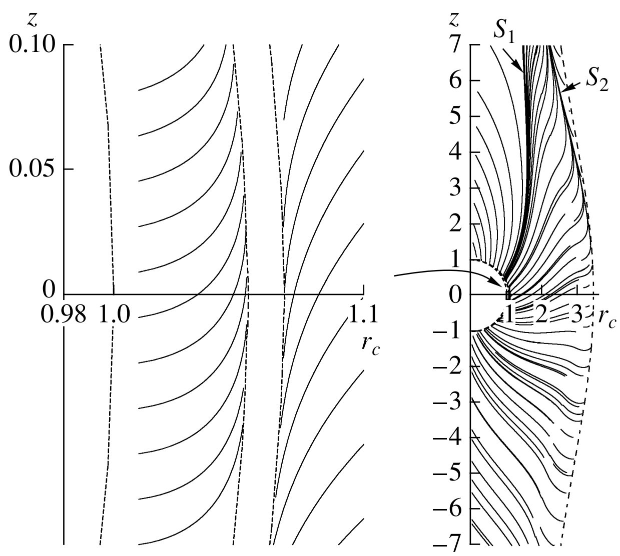

Fig. 2. Nonsymmetric envelopes for $\mu = 0.1$, $\varepsilon = 10$, $\delta = 10^{-3}$. Dashed lines, boundaries of the domain of existence; S_1, S_2, separatrices. On the left, equatorial zone in the vicinity of $r = 1$.

1. for $(1+\varepsilon)\mu > 1$:

$$\frac{2\sqrt{-r_*}}{\sqrt{\varepsilon+1}} \ln \frac{\sqrt{r} + \sqrt{r - r_*}}{\sqrt{r_0} + \sqrt{r_0 - r_*}} = \pm\theta,$$

where

$$r_* = \frac{(\varepsilon+1)(1-\mu)}{1-\mu(\varepsilon+1)}, \quad r_0 = r(0);$$

2. for $(1+\varepsilon)\mu = 1$:

$$2(\sqrt{\mu r} - \sqrt{\mu r_0}) = \pm\theta;$$

3. for $(1+\varepsilon)\mu < 1$:

$$\frac{2\sqrt{r_*}}{\sqrt{\varepsilon+1}} \left(\arcsin \frac{\sqrt{r}}{\sqrt{r_*}} - \arcsin \frac{\sqrt{r_0}}{\sqrt{r_*}} \right) = \pm\theta,$$

and also the family envelope, the spherical envelope

$$r = r_*. \tag{4}$$

In the first case all envelopes are bounded by the sphere (4), in the remaining cases the envelopes do not extend to infinity. In Fig. 1a–1c, the three corresponding types of envelopes are shown in cylindrical coordinates: $r_c = r\sin\theta$, $z = r\cos\theta$. On this plane, all envelopes are perpendicular to the circle $r = 1$, begin and end on either the axis z or the circle $r = 1$, or the envelope curve (4).

B. In the absence of a stellar wind ($\varepsilon = 0$), the shape of the envelope is determined by the algebraic equation

$$\delta \sin^2\theta = f_1(r) / \left(r^3 f_1(r) + 2r^4 f_2(r) \right),$$

which gives double-bottle forms extending to infinity (Fig. 1(d)).

6. NUMERICAL RESULTS

In the general case ($\varepsilon \neq 0$, $\delta \neq 0$), the shape of the envelope is obtained by integrating Eq. (3). Some of the shapes are shown in Fig. 2. As in the nonrelativistic case [4], there exist nonsymmetric bottle-shaped envelopes extending to infinity.

A special feature of the relativistic case is that in the plane $(r_c; z)$ the existence domain has, along with an outer boundary, a crescent-shaped lacuna in the vicinity of $r = 1$. In case $\delta \ll 1$, the distance between the lacuna and the sphere $r = 1$ is of the order δ, and the thickness of the lacuna on the equator is of the order of $\delta^{3/2}\sqrt{(1-\mu)/\varepsilon}$.

A significant peculiarity of the problem is the existence of two separatrices S_1 and S_2 (Fig. 2) which arise in the neighborhood of the equator and go to the infinity along the outer boundary of the existence domain. The separatrices are the most stable forms of envelopes: analogous forms are evidenced, as a rule, in symmetric pairs, in some planetary nebulae (e.g., the Ant, Butterfly and Cat's Eye nebulae [1]).

Stability of the envelopes in the neighborhood of $r = 1$ needs an additional investigation.

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