The inertial lift on a spherical particle settling in a horizontal viscous flow through a vertical slot

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The inertial lift force exerted on a small rigid sphere settling due to gravity in a horizontal channel flow between vertical walls is investigated. The method of matched asymptotic expansions is used to obtain solutions for the disturbance flow on the length scales of the particle radius and the channel width (inner and outer regions, respectively). The channel Reynolds number is finite, while the particle Reynolds numbers that are based on the slip velocity and the mean shear rate are small. The inner flow is described by the linear Stokes equations. The outer problem is governed by the linear Oseen-like equations with the particle effect approximated by a point force. The outer equations are solved numerically using the two-dimensional Fourier transform of the disturbance velocity field. The lift coefficient is evaluated as a function of governing dimensionless parameters: the particle coordinate across the channel, the channel Reynolds number, and the slip parameter. The particle always migrates away from the walls, with an equilibrium position being on the channel centerline. Close to the walls, the lift coefficient is the same regardless of the slip velocity and the channel Reynolds number. At large channel Reynolds numbers, a local maximum of the migration velocity forms near the channel centerline due to a combined effect of the slip, the linear shear, and the curvature of the undisturbed velocity profile. The results obtained are extended to the case when the drag on a particle has components both parallel and perpendicular to the undisturbed flow. One of primary applications of the results is modeling of the cross-flow migration of settling particles during particle transport in a hydraulic fracture. © 2009 American Institute of Physics.
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I. INTRODUCTION

The migration of a single particle settling under gravity at small Reynolds numbers in shear flows of Newtonian fluids was considered in many theoretical studies.[1–13] The inertial lift on a sphere was calculated for different flow configurations, either unbounded or wall bounded, either linear shear or channel flows, and a particle slip velocity directed either along the fluid velocity or along the shear gradient. In the present study, we consider the case when a particle settles in a horizontal viscous flow through a vertical slot (Fig. 1). Apparently, the lift force on a particle in the present geometrical configuration has not been studied before. The problem is important in view of possible applications in chemical engineering and petroleum technology. For example, the flow configuration shown in Fig. 1 corresponds to the transport of proppant particles in a hydraulic fracture, where the cross-flow migration is caused by the lateral force exerted on individual settling particles. It is known that lateral migration of particles may have a significant effect on the overall settling rate.[14] The settling-induced lift force considered in the present study can be used in a model of particle migration across fluid streamlines in a hydraulic fracture. The cross-fracture particle concentration profile may then be incorporated into the models of particle transport, which are implemented into existing commercialized simulators of hydraulic fracturing.[14]

The migration of a particle moving with a nonzero slip velocity in a shear flow of a viscous fluid is due to a small fluid inertia. When the particle Reynolds number, $R_s = aU_s'/\nu$, is small, the inertial term in the Navier–Stokes equations is small compared to the viscous one at distances of the order of the particle radius a from the particle center (in the inner region). Here, U_s' is the dimensional magnitude of the slip velocity (velocity of the sphere center relative to an undisturbed flow velocity at the point), ν is the kinematic viscosity of the fluid, and the prime superscript is used hereinafter to denote the dimensional velocities and spatial coordinates. To the leading order in R_s, the disturbance flow in the inner region is governed by the Stokes equations. However, the inertial term decays with the distance slower than the viscous one. These terms are of the same order at sufficiently large distances from the sphere (in the outer region). For the uniform undisturbed flow, the length scale of the outer region, where the two terms balance, is the Oseen length $L^{Os} = a/R_s = \nu/U_s' \gg a$.

Theoretical studies of particle migration due to the inertial effect were based on solving the Navier–Stokes equations using perturbation methods. A regular perturbation technique can be used when the distance of the particle from the wall is small compared to the length scale of the outer region, i.e., it is assumed that the wall lies within the inner region of the flow. Cox and Brenner[3] considered the migra-

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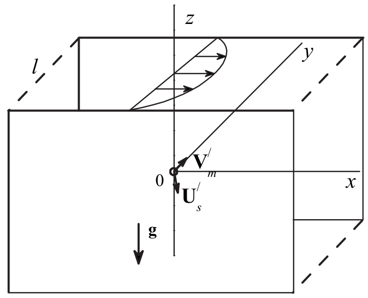

FIG. 1. The sketch of the flow pattern near a particle settling in a viscous flow through a vertical slot.

tion of a non-neutrally buoyant sphere translating in a linear-shear flow at distances from the wall d being large compared to the particle radius, $a \ll d \ll L^{\mathrm{Os}}$. They expressed the lift in terms of Green's functions. The lift was evaluated in linear and parabolic wall-bounded flows.[4–6] The regular perturbations were also applied to the case when $d \sim a$.[10]

For an unbounded linear-shear flow, another length scale of the outer region, $L^{\mathrm{Sa}}=(\nu/G)^{1/2}$, based on the shear rate G, was introduced.[1] Saffman[1] showed that the particle effect in the outer region is equivalent to a point force exerted on the fluid at the center of the sphere. Oseen-like equations governing the outer disturbance flow were solved in terms of Fourier transforms. The disturbance induced by the sphere in the outer region produces a small lateral correction to a uniform flow in the inner region. The migration velocity in the shear direction was calculated for a particle moving parallel to the streamlines of an unbounded flow in the strong-shear limit, when the two small particle Reynolds numbers based on the slip velocity and the shear rate are related as

$$R_s \ll R_p^{1/2} = a(G/\nu)^{1/2} \ll 1. \qquad (1)$$

This inequality means that the linear-shear flow dominates the uniform flow in the outer region. In the case when the sphere moves in an arbitrary direction to the shear flow, other components of the migration velocity were also found for the strong-shear case.[2,12] The lift is zero in an unbounded linear flow when the slip velocity is perpendicular both to the fluid velocity and its gradient, as for the fracture configuration. Asmolov[8] and McLaughlin[9] removed Saffman's restriction (1) on the relative sizes of the two Reynolds numbers and considered a general case of an arbitrary ratio of the Reynolds numbers. They evaluated the migration velocity of a sphere translating parallel to the streamlines of an unbounded shear flow as a function of the slip parameter $\alpha = R_s/R_p^{1/2}$. The velocity is maximum in the strong-shear limit when $\alpha \ll 1$ and is very small, of the order of $\alpha^{-5} \ln \alpha$, in the opposite weak-shear limit, $\alpha \gg 1$.

Hogg[11] and Asmolov[13] studied the inertial migration of a non-neutrally buoyant particle settling in a vertical channel flow. In this case, the migration is due to the following two effects: the Oseen-like inertial migration and the wall effect. The lift coefficient and the particle equilibrium positions were calculated as the functions of the slip parameter and the channel Reynolds number $R_c = U_m' l/\nu$, where l and U_m' are the channel width and the maximum flow velocity. When R_c is large, the wall effect is significant in thin layers near the walls of a width of the order of $lR_c^{-1/2}$.[13] In the core of the flow, excluding near-wall layers, the wall effect is negligible, and the disturbance field in the outer region can be treated as an unbounded one. The effect of the curvature of the velocity profile on the lift is significant.

The goal of the present work is to calculate the migration velocity of a non-neutrally buoyant particle settling in a horizontal viscous flow bounded by vertical impermeable walls. One would expect that the wall effect at large R_c is significant only in the thin layers near the slot walls. In the bulk of the flow, the lift is expected to be small for the configuration considered, since it is zero in an unbounded linear flow. However, it will be shown below that a local maximum of the migration velocity appears at distances of the order of $R_c^{-1/3}$ from the channel center due to a combined effect of the slip, the linear shear, and the curvature of the undisturbed velocity profile. The lift coefficient is of the same order over the entire channel width. This means that the scaling we use to nondimensionalize the lift force is applicable everywhere in the flow domain, both near the walls and in the core of the flow.

The paper is organized as follows. Governing equations of the viscous flow past a small sphere settling in a horizontal slot flow are presented in Sec. II. Solutions in the inner and outer regions are constructed using the method of matched asymptotic expansions in Secs. III and IV, respectively. The outer solution is presented in terms of a two-dimensional Fourier transform. An ordinary differential equation for the Fourier transform of the normal velocity is integrated numerically using an orthonormalization method. Numerical results and a comparison with previous studies are presented in Sec. V. The extension of the approach for arbitrary direction of the slip velocity is presented in Sec. VI. The results are summarized in Sec. VII.

II. GOVERNING EQUATIONS

Consider a rigid sphere settling due to gravity in a flow of a viscous incompressible Newtonian fluid through a vertical slot. The sketch of the flow configuration is shown in Fig. 1. The undisturbed flow in a slot may have not only a horizontal but also a vertical component. The origin of a Cartesian coordinate system is at the center of the sphere and translates with the particle velocity $\mathbf{U}_p'$. Hence, the particle is at rest in this frame of reference. The x-axis is parallel to the slot walls and directed along the undisturbed flow, and y-axis is perpendicular to the walls. The fluid velocity relative to the sphere has generally x-and z-components, U_{sx}' and U_{sz}', so that $U_s' = \sqrt{U_{sx}'^2 + U_{sz}'^2}$. The particle radius a is small compared to the slot width l. The particle Reynolds number and the length scale of the outer region based on the average shear rate, $G_{\mathrm{av}} = U_m'/l$, are introduced by

$$R_p = U_m' a^2/\nu l, \quad L = aR_p^{-1/2} = (\nu l/U_m')^{1/2} = lR_c^{-1/2}.$$

The method of matched asymptotic expansions, based on a small parameter

$$\varepsilon = R_p^{1/2} = a(U_m'/\nu l)^{1/2} \ll 1,$$

is applied to solve the equations governing the disturbance flow. The ratio of the particle size to the channel width and the particle Reynolds numbers based on the particle slip velocity and the local shear rate are also small,

$$a/l = (R_p/R_c)^{1/2} = \varepsilon R_c^{-1/2} \ll 1, \quad R_s \sim \varepsilon \ll 1.$$

The other dimensionless groups, $U_{sx}=U_{sx}'/U_m'$, $U_{sz}=U_{sz}'/U_m'$, d/l, R_c are taken to be of the order of unity, where d is the distance of the particle to the nearest wall. Flow disturbances due to drag on the particle dominate those due to shear and particle rotation, when $U_s \gg \varepsilon^2$.[11]

The asymptotic analysis follows the study[13] of the inertial migration in a vertical channel for the case when the slip velocity is parallel to the undisturbed flow. The novelty of the present study is that the undisturbed velocity of the fluid in the absence of the sphere has two components,

$$\mathbf{v}' = v_x'\mathbf{e}_x + v_z'\mathbf{e}_z,$$

$$v_x' = U_{sx}' + U_m'\left(\gamma\frac{y'}{l} - 4\frac{y'^2}{l^2}\right), \quad v_z' = U_{sz}',$$

where $\mathbf{e}_x$ and $\mathbf{e}_z$ are the unit vectors of the x- and z-axes, respectively, and $\gamma=4-8d/l$ is the dimensionless shear rate at the particle center. To the leading order in ε, the particle translates parallel to the walls, and its migration velocity in the normal direction due to the fluid inertia and the wall effect is small compared to the slip velocity. Hence, the problem can be treated as a quasisteady one.

The dimensionless variables are introduced by the formulas

$$\mathbf{r} = \mathbf{r}'/a, \quad \mathbf{u} = \mathbf{u}'/U_m', \quad \mathbf{\Omega}_p = \mathbf{\Omega}_p' l/U_m', \quad \mathbf{F}_p = \mathbf{F}_p'/\mu a U_m',$$

where $\mathbf{u}$ is the disturbance of the fluid velocity and $\mathbf{\Omega}_p$ and $\mathbf{F}_p$ are the dimensionless rotational velocity and the force on the particle, respectively. The dimensionless undisturbed velocity field is given by

$$\mathbf{v} = \mathbf{v}'/U_m' = (U_{sx} + \varepsilon\gamma y R_c^{-1/2} - 4\varepsilon^2 y^2 R_c^{-1})\mathbf{e}_x + U_{sz}\mathbf{e}_z. \quad (2)$$

We first consider in detail the case when the particle settles under gravity in a horizontal undisturbed flow, so that the slip velocity is perpendicular to the fluid velocity. Then the x-component of the force on the particle is zero. It follows from Faxen formula that the corresponding slip velocity is $U_{sx}=O(\varepsilon^2)$, and, to the first order, may be neglected. The extension of the analysis to the case $U_{sx}\neq 0$ is presented in Sec. VI.

The dimensionless Navier–Stokes equations governing the disturbance flow are written as

$$\varepsilon R_c^{1/2}[(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v}] = -\nabla p + \nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = 0. \quad (3)$$

The boundary conditions are specified as follows. The no-slip condition is prescribed on the particle surface and the channel walls, and it is assumed that the velocity disturbance decays far from the particle,

$$\mathbf{u} = -\mathbf{v} + \varepsilon R_c^{-1/2}\mathbf{\Omega}_p\times\mathbf{r} = -U_{sz}\mathbf{e}_z + \varepsilon R_c^{-1/2} \times(\mathbf{\Omega}_p\times\mathbf{r} - \gamma y\mathbf{e}_x) + 4\varepsilon^2 y^2 R_c^{-1}\mathbf{e}_x \quad \text{on } r=1,$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } y = -\varepsilon^{-1}R_c^{1/2}d/l, \quad \varepsilon^{-1}R_c^{1/2}(1-d/l), \quad (4)$$

$$\mathbf{u}\to\mathbf{0} \quad \text{as } r\to\infty.$$

III. INNER SOLUTION

We seek the dimensionless disturbance velocity and pressure fields in the form of asymptotic series in ε,

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + o(\varepsilon), \quad p = p_0 + \varepsilon p_1 + o(\varepsilon).$$

Substituting these expansions into Eqs. (3) and (4) in the leading-order approximation, we obtain the Stokes equations in an unbounded flow,

$$\nabla^2\mathbf{u}_0 - \nabla p_0 = 0, \quad (5)$$

$$\nabla\cdot\mathbf{u}_0 = 0, \quad (6)$$

$$\mathbf{u}_0 = -U_{sz}\mathbf{e}_z \quad \text{on } r=1, \quad (7)$$

$$\mathbf{u}_0\to\mathbf{0} \quad \text{as } r\to\infty. \quad (8)$$

It should be noted that the boundary condition (7) involves the uniform velocity only. The terms in Eq. (4) corresponding to the shear rate and the particle angular velocity are of order ε, and the term due to the curvature of the undisturbed velocity profile is of order ε^2. Hence, these terms should be omitted in the leading-order approximation. The solution of Eqs. (5)–(8) is the well-known Stokes solution,

$$\mathbf{u}_0 = -U_{sz}\left[\mathbf{e}_z\left(\frac{3}{4r} + \frac{1}{4r^3}\right) + \frac{3}{4}\frac{z\mathbf{r}}{r^2}\left(\frac{1}{r} - \frac{1}{r^3}\right)\right].$$

This axisymmetric solution gives the drag on the sphere, $\mathbf{F}_p=6\pi U_{sz}\mathbf{e}_z$, and yields no lateral force or torque. The velocity field decays with distance from the sphere as r^{-1},

$$\mathbf{u}_0\to\mathbf{u}^S \quad \text{as } r\to\infty,$$

$$\mathbf{u}^S = -\frac{3}{4}U_{sz}\left(\frac{\mathbf{e}_z}{r} + \frac{z\mathbf{r}}{r^3}\right).$$

The Stokeslet velocity field $\mathbf{u}^S$ corresponds to the viscous flow driven by the point force,

$$\mathbf{F}_f = F_f\mathbf{e}_z = -\mathbf{F}_p = -6\pi U_{sz}\mathbf{e}_z.$$

Collecting the terms of order ε in the asymptotic expansions of Eqs. (3) and (4) and taking into account Eq. (2), we derive the equations governing the first-order inner solution,

$$\nabla^2\mathbf{u}_1 - \nabla p_1 = R_c^{1/2}[(\mathbf{u}_0 + U_{sz}\mathbf{e}_z)\cdot\nabla]\mathbf{u}_0, \quad (9)$$

$$\nabla\cdot\mathbf{u}_1 = 0, \quad (10)$$

$$\mathbf{u}_1 = R_c^{-1/2}(\mathbf{\Omega}_p\times\mathbf{r} - y\gamma\mathbf{e}_x) \quad \text{on } r=1. \quad (11)$$

Note that even to this order of ε, the curvature of the undisturbed velocity profile does not enter into the inner-flow equations. Since Eqs. (9)–(11) are linear, the solution of these equations can be sought as the superposition of three terms,[13]

$$\mathbf{u}_1 = R_c^{1/2}\mathbf{u}^{PP} + \mathbf{u}^{nb} + \mathbf{w}. \quad (12)$$

The boundary conditions on the particle surface for the three terms are

$$\mathbf{u}^{PP} = \mathbf{0}, \quad \mathbf{u}^{nb} = R_c^{-1/2}(\mathbf{\Omega}_p \times \mathbf{r} - y\gamma\mathbf{e}_x),$$

$$\mathbf{w} = \mathbf{0} \quad \text{on} \quad r=1.$$

The first term in Eq. (12) is the solution of inhomogeneous equations (9) and (10) for a uniform undisturbed flow obtained by Proudman and Pearson,[15]

$$\mathbf{u}^{PP} = \frac{3}{32}U_{sz}^2\left[\left(2 - \frac{3}{r} + \frac{1}{r^2} - \frac{1}{r^3} + \frac{1}{r^4}\right)\left(1 - \frac{3z^2}{r^2}\right)\frac{\mathbf{r}}{r} + \left(4 - \frac{3}{r} + \frac{1}{r^3} - \frac{2}{r^4}\right)\left(\frac{z^2\mathbf{r}}{r^3} - \frac{z\mathbf{e}_z}{r}\right)\right].$$

Two other terms satisfy homogeneous Stokes equations (5) and (6). The term $\mathbf{u}^{nb}$ is the solution for a force- and torque-free (neutrally buoyant) particle in an unbounded linear-shear flow. It accounts for the effect of particle rotation in a shear flow. The last term, $\mathbf{w}(\mathbf{r})$, is the solution of Eqs. (5) and (6), which tends to a uniform velocity $\mathbf{w}_\infty$ at infinity. The first-order solution $\mathbf{u}_1$ does not decay with r, and the boundary condition at infinity has to be replaced by a matching condition with the outer flow. The outer limit of the inner solution is

$$\mathbf{u}_1|_{r\to\infty} = \mathbf{u}_\infty^{PP}(\mathbf{r}/r) + \mathbf{w}_\infty,$$

$$\mathbf{u}_\infty^{PP} = \mathbf{u}^{PP}|_{r\to\infty} = \frac{3}{16}U_{sz}^2\left[\left(1 - \frac{3z^2}{r^2}\right)\frac{\mathbf{r}}{r} + 2\left(\frac{z^2\mathbf{r}}{r^3} - \frac{z\mathbf{e}_z}{r}\right)\right]. \quad (13)$$

The first two terms in Eq. (12) give no contribution to the migration velocity because of the symmetry. Hence, the migration is due to the transverse component of the velocity $\mathbf{w}_\infty$, which should be found from the matching condition with the outer flow. The migration velocity of an inertialess particle is $V_m = \varepsilon w_{y\infty}$.

IV. OUTER SOLUTION

The outer coordinates are introduced by the scaling $\mathbf{R}=(X,Y,Z)=\varepsilon\mathbf{r}=\mathbf{r}'/L$. The velocity and pressure in the outer region can be presented as

$$\mathbf{u} = \varepsilon\mathbf{U} + o(\varepsilon), \quad \mathbf{U} = (U_x, U_y, U_z), \quad p = \varepsilon^2 P + o(\varepsilon^2).$$

As the disturbances are small compared to the undisturbed flow in the outer region, the inertial terms in the Navier–Stokes equations (3) can be linearized,

$$\varepsilon R_c^{1/2}[(\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v}] = \varepsilon^2[(\mathbf{V}\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{V}] + O(\varepsilon^3).$$

Here, the components of the undisturbed flow velocity are written in terms of the outer coordinates as follows:

$$\mathbf{V} = (V_x, 0, V_{sz}), \quad V_x = \gamma Y - 4R_c^{-1/2}Y^2, \quad V_{sz} = U_{sz}R_c^{1/2}. \quad (14)$$

The slip parameter V_{sz} characterizes the relative size of the slip velocity and the linear-shear term in the undisturbed velocity. Hence, this parameter is analogous to the ratio of the particle Reynolds numbers $\alpha=R_s/R_p^{1/2}$ used in the studies of particle migration in linear-shear flows.[8,9] The strong-shear limit corresponds to the case $\varepsilon^2 \ll V_{sz} \ll 1$.

The matching condition with the inner region can be satisfied if the point force $\mathbf{F}_f$ is introduced into the momentum equation.[1] Therefore, the equations governing the outer field are reduced to the linear Oseen-like equations,

$$V_x\frac{\partial\mathbf{U}}{\partial X} + V_{sz}\frac{\partial\mathbf{U}}{\partial Z} + \frac{dV_x}{dY}U_y\mathbf{e}_x + \nabla P - \nabla^2\mathbf{U} = -6\pi U_{sz}\delta(\mathbf{R})\mathbf{e}_z,$$

$$\nabla\cdot\mathbf{U} = 0, \quad (15)$$

$$\mathbf{U} = 0 \quad \text{on} \quad Y = -R_c^{1/2}d/l, \quad R_c^{1/2}(1-d/l),$$

$$\mathbf{U} = 0 \quad \text{as} \quad R \to \infty.$$

To solve the outer-flow equations, the two-dimensional Fourier transforms $\mathbf{U}^*$, P^* of the disturbance field are introduced as

$$\begin{Bmatrix}\mathbf{U}^*(k_x,Y,k_z)\\ P^*(k_x,Y,k_z)\end{Bmatrix} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}\mathbf{U}\\ P\end{Bmatrix} \times\exp[-i(k_xX + k_zZ)]dXdZ.$$

The velocity and pressure fields are then given by the inverse Fourier transforms,

$$\begin{Bmatrix}\mathbf{U}\\ P\end{Bmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}\mathbf{U}^*\\ P^*\end{Bmatrix}\exp[i(k_xX + k_zZ)]dk_xdk_z. \quad (16)$$

The Oseen-like equations (15) are rewritten in terms of the Fourier transforms as follows:

$$(ik_xV_x + ik_zV_{sz})\mathbf{U}^* + \frac{dV_x}{dY}U_y^*\mathbf{e}_x + \nabla^*P^* - \Delta^*\mathbf{U}^* = -\frac{3}{2\pi}U_{sz}\delta(Y)\mathbf{e}_z,$$

$$\nabla^*\cdot\mathbf{U}^* = 0, \quad (17)$$

$$\nabla^* = \left(ik_x, \frac{d}{dY}, ik_z\right), \quad \Delta^* = \frac{d^2}{dY^2} - k^2, \quad k^2 = k_x^2 + k_z^2,$$

$$\mathbf{U}^* = 0 \quad \text{on} \quad Y = -R_c^{1/2}d/l, \quad R_c^{1/2}(1-d/l).$$

Thus we obtain the system of linear ordinary differential equations with respect to Y. Multiplying the x-component of the momentum equation (17) by ik_x, the z-component by ik_z, adding the resulting equations, and using the equation of continuity, one obtains the value of P^* as

$$P^* = \frac{1}{k^2}\left[(\Delta^* - ik_xV_x - ik_zV_{sz})\frac{dU_y^*}{dY} + ik_x\frac{dV_x}{dY}U_y^* + \frac{3}{2\pi}ik_zU_{sz}\delta(Y)\right].$$

Differentiating the last equation with respect to Y, substituting the resulting expression into the y-component of the momentum equation, one can derive a single ordinary differential equation governing the Fourier transform of the lateral velocity,

$$[\Delta^{*2} - (ik_xV_x + ik_zV_{sz})\Delta^* - ik_x 8R_c^{-1/2}]U_y^* = -\frac{3}{2\pi}ik_zU_{sz}\frac{d\delta(Y)}{dY},$$

$$U_y^* = \frac{dU_y^*}{dY} = 0 \quad \text{on} \quad Y = -R_c^{1/2}d/l, \quad R_c^{1/2}(1-d/l). \tag{18}$$

The term on the right-hand side of Eq. (18) is equivalent to a jump condition for the second derivative at the origin of the coordinate system,

$$\left[\frac{d^2U_y^*}{dY^2}\right] = -\frac{3}{2\pi}ik_zU_{sz},$$

where $[f]=f(+0)-f(-0)$ is the magnitude of the jump. The function U_y^* and its first and third derivatives are continuous at the origin. These features of U_y^* can be proven if one integrates Eq. (18) successively four times over a small interval $(-\Delta Y, \Delta Y)$, $\Delta Y \to 0$, and solves a linear system for $[d^{(i)}U_y^*/dY^{(i)}]$, $i=0-3$.

Matching of the outer field back to the inner flow requires the outer limit of the two-term inner expansion, $\mathbf{u}_0+\varepsilon\mathbf{u}_1$, to be equal to the outer solution at the origin. As a result one has for the outer limit of the first-order term Eq. (13),[1]

$$\mathbf{u}_1|_{r\to\infty} = (\mathbf{U} - \mathbf{U}^S)|_{R\to 0},$$

where $\mathbf{U}^S$ is the Stokeslet velocity field written in terms of the outer coordinates,

$$\mathbf{U}^S = -\frac{3}{4}U_{sz}\left(\frac{\mathbf{e}_z}{R} + \frac{Z\mathbf{R}}{R^3}\right),$$

which matches with the outer limit of $\mathbf{u}_0$.

Small distances R correspond to large k, and the matching condition means that the numerical solution of Eq. (18) should be equal to the two-dimensional Fourier transform of the Stokeslet at large Fourier numbers,

$$U_y^* = U_y^{S*}[1 + O(k^{-1})] \quad \text{as} \quad k \gg 1,$$

where U_y^{S*} is given by[7]

$$U_y^{S*} = \frac{3}{8\pi}\frac{ik_x}{k}U_{sz}Y\exp(-k|Y|).$$

The migration velocity V_m of an inertialess particle is expressed in terms of the Fourier transform of the lateral velocity at the origin as[1]

$$V_m = \varepsilon\,\mathrm{Re}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(U_y^* - U_y^{S*})|_{Y\to 0}dk_xdk_z\right]. \tag{19}$$

Thus, the problem is reduced to finding $U_y^*(k_x,0,k_z)$ from the numerical solution of Eq. (18). The numerical integration of the finite-difference version of Eq. (18) presents a certain difficulty at large R_c. The routine technique fails to converge in this case, since it does not permit to resolve properly all the linearly independent solutions of the ordinary differential equation. Therefore, the technique does not allow one to calculate the lift for channel Reynolds numbers larger than approximately 100.[11] The problem is eliminated in the present work similar to the study[13] using the orthonormalization method,[16] which makes it possible to integrate Eq. (18) up to $k_{\max}=128$ for any R_c. For large k, the asymptotic approximation of the Fourier transform similar to that of the work[11] can be obtained,

$$U_y^*(k_x,0,k_z) - U_y^{S*}(k_x,0,k_z) = \gamma U_{sz}\frac{3}{16\pi}\frac{k_xk_z}{k^5} + O(k^{-5}) \quad \text{as} \quad k\to\infty. \tag{20}$$

The first term on the right side of Eq. (20) is an odd function of k_x and k_z, and its integral over k_x is zero. For this reason, the contribution of large k to Eq. (19) is small, of the order of $k_{\max}^{-3}$. The numerical integration of Eq. (19) is fulfilled using the plane polar coordinates k, $\theta=\arccos(k_x/k)$ instead of k_x, k_z. It can be verified directly from Eq. (18) that

$$U_y^*(k_x,Y,k_z) = \overline{U_y^*}(-k_x,Y,-k_z),$$

where the overbar denotes the complex conjugated value. Hence, it is sufficient to integrate Eq. (19) over the first and second quadrants.

V. NUMERICAL RESULTS

The lift is obtained numerically as a function of the three main dimensionless groups: the particle position across the slot d/l, the slip parameter V_{sz}, and the channel Reynolds number R_c. For the sake of illustration, we consider wide ranges of variation of V_{sz} and R_c, which cover the magnitudes relevant to fracturing applications: $0.001 \le V_{sz} \le 4$, $0.1 \le R_c \le 1000$. The results are compared with the predictions of the work[5] for a small sphere settling between parallel walls in a stagnant fluid. Vasseur and Cox[5] considered the case when the distances from the walls are also much larger than the sphere radius but still smaller than the outer-region length scale, so the walls lie within the inner region. Then the migration velocity is given by[5]

$$V_m^{\mathrm{VC}} = R_sU_{sz}c_m^{\mathrm{VC}}.$$

The lift coefficient $c_m^{\mathrm{VC}}(d/l)$ can be approximated by the polynomial

$$c_m^{\mathrm{VC}} = -0.3125\varsigma - 0.0996\varsigma^3 + 14.34\varsigma^5 - 116.3\varsigma^7 + 403.7\varsigma^9 - 518\varsigma^{11},$$

$$\varsigma = d/l - 1/2.$$

The lift is directed away from the walls. The maximum value of the lift coefficient is attained on the wall: $c_m^{\mathrm{VC}}(d/l \to 0) = 3/32$. The particle equilibrium position where the lift vanishes is on the channel axis, at $d/l = 0.5$.

For a particle settling in a horizontal channel flow, we found that the migration velocity should be scaled in a similar manner,

$$V_m = R_p^{1/2} U_{sz} V_{sz} c_m = R_s U_{sz} c_m, \tag{21}$$

or, in the dimensional form,

$$V_m' = \frac{U_{sz}'^2 a}{\nu} c_m(d/l, V_{sz}, R_c).$$

The quadratic dependence of the migration velocity on the slip velocity differs from the linear scaling obtained for the case when the slip velocity is parallel to the undisturbed flow.[13] This quadratic scaling can be deduced from the behavior of the Fourier transform of the lateral velocity, which, in turn, follows from the solution of Eq. (18) for the strong-shear case, $V_{sz} \ll 1$. The equation is linear; hence the solution is proportional to the right-hand side, $U_y^* \propto k_z U_{sz}$. The slip parameter V_{sz} enters into the term in the round brackets on the left-hand side multiplied by k_z. All other terms are even functions of k_z as they involve k_x and k^2 only. For this reason, the solution can be sought in the form of series in $V_{sz} k_z$ when the slip parameter is small,

$$U_y^* = k_z U_{sz}[\psi_0(k_x, k^2) + V_{sz} k_z \psi_1(k_x, k^2) + \cdots] \quad \text{as} \quad V_{sz} \ll 1.$$

The leading-order term in the expansion is linear in k_z and, hence, its contribution to the integral (19) is zero. It also follows from the inverse Fourier integral (16) that the corresponding velocity field is an odd function of Z. Hence, the first-order term, $k_z^2 U_{sz} V_{sz} \psi_1$, only contributes to Eq. (19). As a result, $V_m \propto U_{sz} V_{sz}$ as $V_{sz} \ll 1$, and we obtain the scaling (21) for the migration velocity.

The lift coefficient $c_m(d/l, V_{sz}, R_c)$ is shown in Fig. 2 as a function of the cross-channel particle coordinate for different V_{sz} and R_c in comparison with $c_m^{\mathrm{VC}}(d/l)$.[5] The curves are presented only for a half of the channel $(0 \le d/l \le 0.5)$ since they are antisymmetric with respect to the channel centerline.

The lift coefficient is positive for all values of the governing dimensionless parameters, i.e., the particle always migrates from the walls toward the channel centerline. Near the walls, the lift coefficient coincides with the predictions by Vasseur and Cox[5] given by $c_m(d/l \to 0, V_{sz}, R_c) = 3/32$. The migration velocities, calculated for different small (but nonzero) values of the slip parameter, $0.001 \le V_{sz} \le 0.1$ and scaled using Eq. (21), tend to the same dependence (solid curves) in the limit $V_{sz} \to 0$ (the strong-shear case). For $R_c < 1$, the undisturbed flow affects the migration only slightly. It can be seen in Fig. 2(a) that the dependence of the lift coefficient on the particle position calculated for different slip parameters, $0.001 \le V_{sz} \le 4$, is very close to the results for migration of a particle settling in a stagnant fluid.[5] The lift coefficient decreases with the distance from the wall when $R_c \le 10$ [Figs. 2(a) and 2(b)]. At large R_c, the wall effect manifests itself only in the thin layers near the walls at distances of the order of the outer-region length scale, $L = l R_c^{-1/2}$.

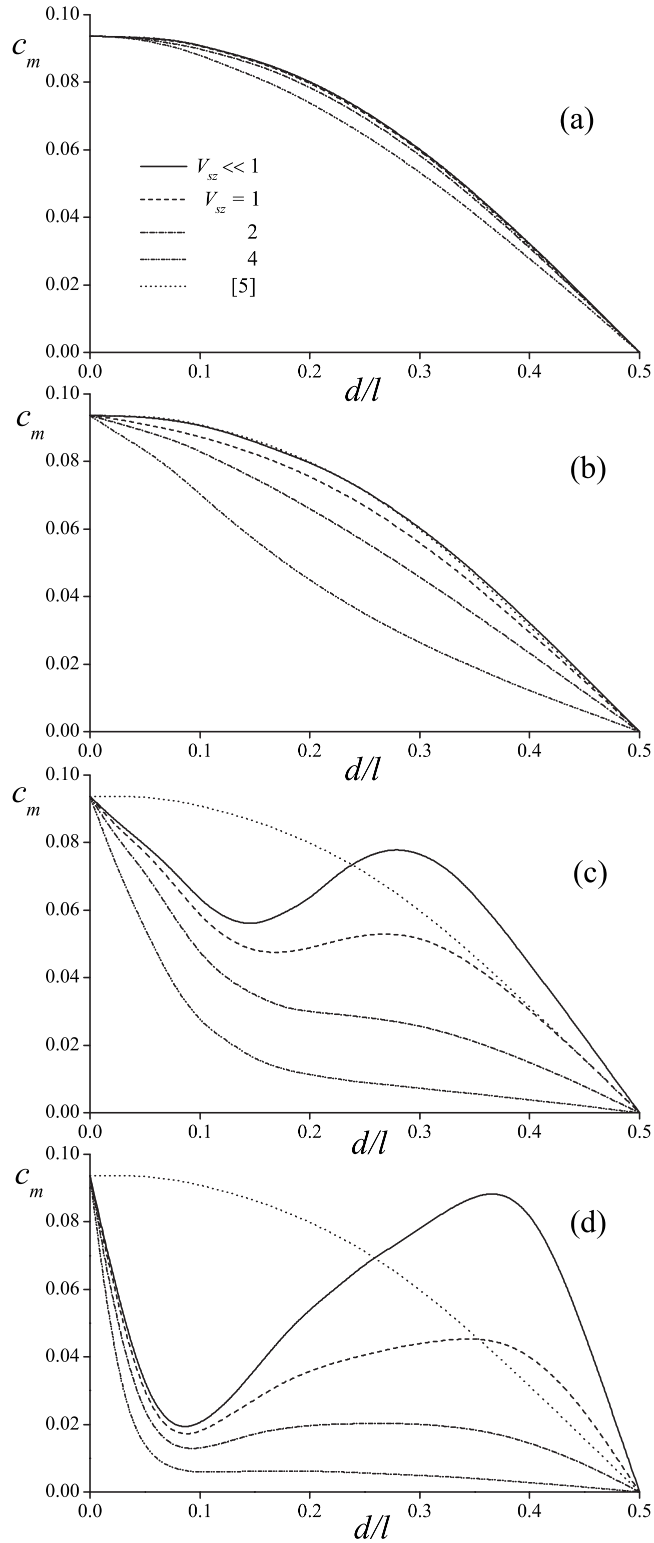

FIG. 2. Lift coefficient vs particle coordinate across the slot and comparison with the predictions by Vasseur and Cox (Ref. 5): (a) $R_c = 1$, (b) $R_c = 10$ (c) $R_c = 100$, and (d) $R_c = 1000$.

One would expect that the migration velocity is small at $R_c \gg 1$ in the core of the flow where the wall effect is negligible since it is zero in an unbounded linear flow.[2] However, this is valid only when $V_{sz} \geq 2$. At smaller slip parameters and large channel Reynolds numbers, c_m is finite in the core of the flow, and an extra maximum forms near the channel centerline [see solid and dashed curves in Figs. 2(c) and 2(d)]. The origin of the extra maximum cannot be explained by the wall effect, which vanishes with the increase in the distance from the wall; also, the shear rate is small near the axis. However, the curvature of the undisturbed profile is finite and another length scale of the outer region can be introduced based on the curvature, $L^c = lR_c^{-1/3}$, and Eq. (18) governing the Fourier transform of the lateral velocity is then rewritten as

$$[\Delta^{c*2} - (ik_x^c V_x^c + ik_z^c V_{sz}^c)\Delta^{c*} - 8ik_x^c]U_y^{c*} = -\frac{3}{2\pi} ik_z^c U_{sz} \frac{d\delta(Y^c)}{dY^c}, \qquad (22)$$

$$U_y^{c*} = \frac{dU_y^{c*}}{dY^c} = 0 \quad \text{on} \quad Y^c = \pm R_c^{1/3}/2 - \xi,$$

$$V_x^c = 4(2\xi Y^c - Y^{c2}), \quad V_{sz}^c = U_{sz} R_c^{2/3}, \qquad (23)$$

$$\xi = (1/2 - d/l)R_c^{1/3}, \quad Y^c = y'/L^c = R_c^{-1/6} Y, \quad \mathbf{k}^c = R_c^{1/6}\mathbf{k}.$$

When the new slip parameter V_{sz}^c is fixed, Eq. (22) does not involve R_c. The distances to the walls scaled by L^c are large when R_c is large and the particle is close to the channel center (the stretched distance ξ is of order unity). This means that the wall effect is negligible in this region and the outer flow can be treated as an unbounded parabolic flow. As a result, the solution of Eq. (22) and the lift coefficient scarcely depend on R_c. Figure 3 presents c_m as a function of ξ obtained for given V_{sz}^c and various channel Reynolds numbers: $R_c = 100, 300, 1000$. The dependences calculated for different R_c are close to each other when $\xi < 1$. Thus, the lift near the axis at large R_c is due to the combination of the slip and the unbounded linear and quadratic shear rates. The migration velocity is linear in ξ when $\xi \ll 1$. The migration velocity reaches the maximum value when $\xi \approx 1$, i.e., when the linear and quadratic terms in the undisturbed profile (23) become comparable.

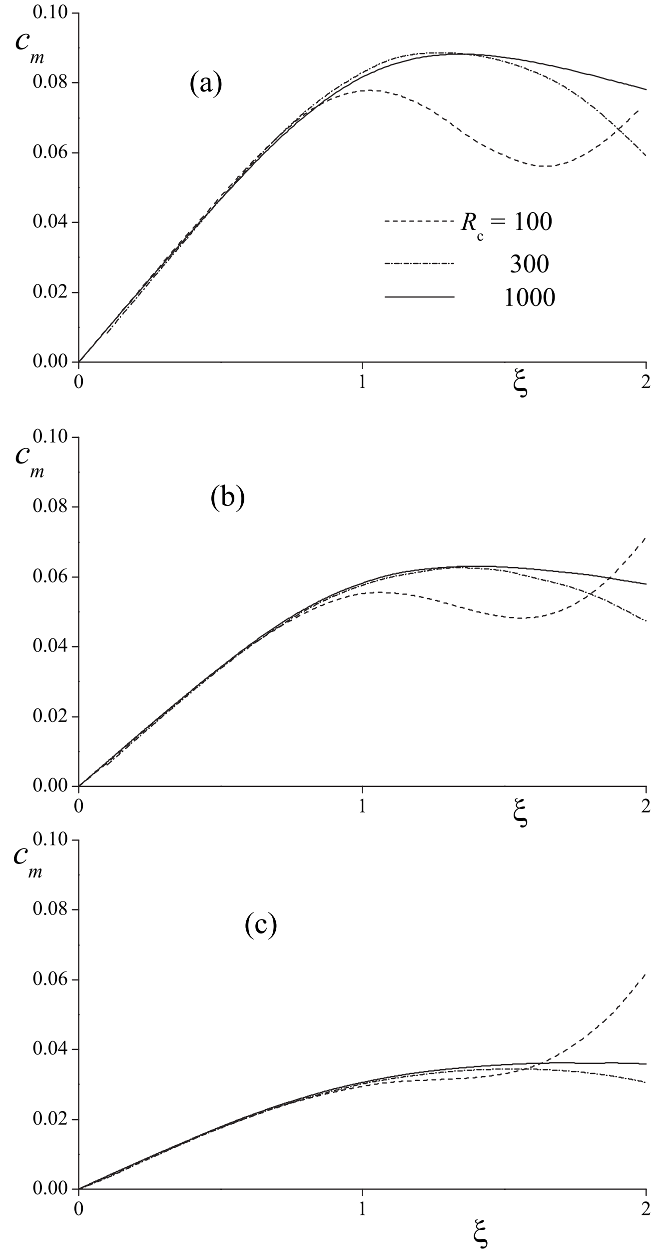

FIG. 3. Lift coefficient near the channel centerline at large R_c vs dimensionless distance from the axis $\xi = (1/2 - d/l)R_c^{1/3}$. (a) Strong-shear limit: $V_{sz}^c \ll 1$, (b) $V_{sz}^c = 2$, and (c) $V_{sz}^c = 4$.

VI. ARBITRARY DIRECTION OF THE SLIP VELOCITY

We have considered above the inertial migration of a particle settling in a horizontal flow in a vertical slot when the slip velocity is perpendicular to the Poiseuille flow of the carrier fluid. The restriction can be removed and the results obtained can be extended to the case when the drag on a particle has both horizontal and vertical components,

$$\mathbf{F}_p = -\mathbf{F}_f = F_x \mathbf{e}_x + F_z \mathbf{e}_z = 6\pi(U_{sx}\mathbf{e}_x + U_{sz}\mathbf{e}_z). \qquad (24)$$

Hogg[11] and Asmolov[13] studied the case when the drag on a sphere is directed along the fluid flow, $\mathbf{F}_p = F_x \mathbf{e}_x = 6\pi U_{sx} \mathbf{e}_x$. The migration velocity is linear in the slip velocity,

$$V_m = R_p^{1/2} U_{sx} c_m^x (d/l, U_{sx}, R_c). \qquad (25)$$

The results obtained numerically in the present work and in the studies[11,13] can be combined for the strong-shear case. Solution in the inner region is constructed in the general case similar to Sec. III. The leading-order term and the first term in the first-order solution (12) are again the solutions obtained by Stokes, $\mathbf{u}_0$, and by Proudman and Pearson,[15] $\mathbf{u}^{PP}$, respectively, but corresponding to another uniform undisturbed velocity, $U_{sx}\mathbf{e}_x + U_{sz}\mathbf{e}_z$. The second term in Eq. (12) is the same and accounts for the effect of particle rotation in a shear flow. All the terms give no contribution to the lateral force because of the symmetry, and the migration is again due to the transverse component of the velocity $\mathbf{w}_\infty$, which is found from the matching condition with the outer flow. The particle effect on the outer disturbance flow is equivalent to the point drag force on the particle, and Eq. (15) can be rewritten with the right side followed from Eq. (24): $-6\pi(U_{sx}\mathbf{e}_x + U_{sz}\mathbf{e}_z)\delta(\mathbf{R})$. Since the Oseen-like equations (15) are linear, the numerical solutions corresponding to each drag component can be constructed separately. However, the slip velocities also enter into the expression for the velocity of the undisturbed flow Eq. (14), which should be written for the general case as

$$\mathbf{V} = (V_{sx} + \gamma Y - 4R_c^{-1/2}Y^2)\mathbf{e}_x + V_{sz}\mathbf{e}_x,$$

$$V_{sx} = U_{sx}R_c^{1/2}.$$

Hence, the solutions corresponding to each drag component will depend on both V_{sx} and V_{sz} when the slip velocities are finite. However, for the strong-shear case, $|V_{sx}| \ll 1$, $|V_{sz}| \ll 1$, the effect of V_{sx} on the part of the solution proportional to the vertical drag F_z can be neglected and vice versa. Then, combining Eqs. (21) and (25), we obtain the following formula for the migration velocity:

$$V_m = R_p^{1/2}(U_{sx}c_m^x + U_{sz}V_{sz}c_m^z) \quad \text{as} \quad |V_{sx}| \ll 1, |V_{sz}| \ll 1. \tag{26}$$

It should be noted that the two terms in Eq. (26) can be sizable even for the case when the x-component of the slip velocity is small compared to the z-component $|U_{sx}| \ll |U_{sz}| \ll 1$. The reason is that the first term in Eq. (26) is linear in the slip velocity, while the second one is quadratic.

The cross-flow migration of particles in dilute-suspension flows is important in a variety of chemical engineering and oilfield applications, e.g., hydraulic fracturing. The results of the present study can be used in the existing continuum models[17–19] of suspension transport through a fracture. Usually, these models are obtained under the assumption that the cross-flow particle concentration profile is uniform. However, it is known that the migration of particles across fluid streamlines can lead to the formation of a central sheet of concentrated suspension with particle-free layers near the walls.[14,20] As a result, the rate of particle sedimentation to the fracture bottom increases significantly as compared to that for a uniformly dispersed suspension. In order to model this phenomenon, the lift force obtained in the present work can be now included into a continuum model of particle migration in a dilute-suspension flow through a vertical slot. The inclusion of a nonuniform cross-flow concentration profile resulting from the inertial migration of particles will improve existing models of particle transport in fractures implemented into commercial simulators of hydraulic fracturing.

VII. CONCLUSIONS

The inertial migration of a small rigid spherical particle settling under gravity in a horizontal flow of a Newtonian fluid through a vertical slot is investigated. Matched asymptotic expansions are applied to find the solution of the Oseen-like equations governing the outer disturbance flow past a particle on the length scale of the channel width and the solution of the Stokes equations governing the inner flow on the length scale of the particle radius. The problem is reduced to a fourth-order ordinary differential equation for the two-dimensional Fourier transform of the lateral velocity, which is solved numerically using the orthonormalization method. The migration velocity is calculated as a function of the three governing dimensionless parameters: the particle distance to the closest wall d/l, the channel Reynolds number R_c, and the slip parameter V_{sz}.

The migration velocity scales like $V_m' = U_{sz}'^2 ac_m/\nu$ and is always positive, i.e., the particle migrates away from the walls with an equilibrium position being on the channel centerline. The lift coefficient c_m is the same close to the wall regardless of the values of R_c and V_{sz}. It decreases with the distance from the wall when $R_c \le 10$. At larger channel Reynolds numbers, a local maximum of the migration velocity forms at distances of the order of $R_c^{-1/3}$ from the channel centerline, which is due to a combined effect of the slip, the linear shear, and the curvature of the undisturbed velocity profile. Extension of the results is obtained for the case when the drag on a particle has components both parallel and perpendicular to the undisturbed flow.

The settling-induced lift force obtained in the present work can be then used in a continuum model of particle migration in a dilute-suspension flow through a vertical slot. A cross-flow particle concentration profile resulting from migration can be further used in models of particle transport through a slot, which are being developed for various oilfield or chemical engineering applications, e.g., for mathematical modeling of hydraulic fracturing.

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